Inflection points of bisoptic curves of conics

Th. Dana-Picard

Jerusalem College of Technology, ndp@jct.ac.il

Let be given a plane curve $\mathscr{C}$ and an angle θ. If it exists, the geometric locus of points through which passes a pair of tangents to $\mathscr{C}$ making an angle equal to θ is called an *isoptic curve* of $\mathscr{C}$. The name comes from the fact that from points on this geometric locus the curve $\mathscr{C}$ is seen under an angle equal to θ. If $\mathscr{C}$ is an ellipse and $\theta = 90^o$, the isoptic curve is the so-called director circle of the ellipse. The study of isoptic curves has been an active field of research for along time, both for strictly convex curves and for open curves; see for example [1], [7], [8].

With the developments of Computer Algebra Systems (CAS) and of Dynamical Geometry Systems (DGS), the study of isoptic curves has found new energies. In [2] and [3], it has been shown that if $\mathscr{C}$ is either an ellipse or a hyperbola, for a non-right angle, the isoptic curve is Spiric of Perseus (also called Oval of Cassini) (see [9], i.e. the intersection of a torus with a plane parallel to the axis of the torus; see Figure 1.

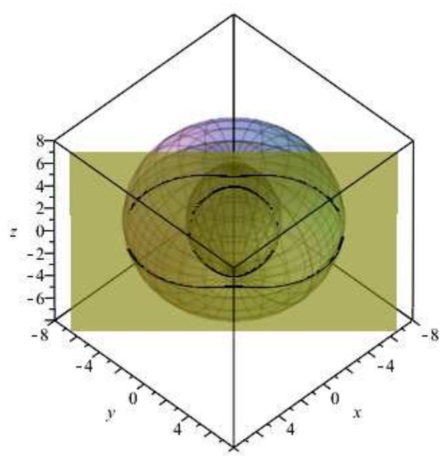

Figure 1: A spiric curve

Technology is used both to visualize the geometric situation and to solve the systems of equations yielded by the algebraic translation of the geometric data. For conic sections, the equations which have been obtained are non linear polynomial equations. The systems of equations have been solved using algorithms based on

Gröbner bases computations; for this the polynomials are viewed as generating ideals in a polynomial ring. Partial results are obtained as a parametric representation of a curve, then implicitization is performed using similar techniques. A noticeable feature we have to deal with is that the curve may not be given by a single parametrization, but is rather presented as the union of numerous parameterized arcs.

It must be mentioned that in order to obtain polynomial equations, squaring both sides is often used. The computations provide at the same time isoptic curves for angle θ and for $180^o - \theta$., whence the name *bisoptic* curves.

In this case, the bisoptic curve is the intersection of a self-intersecting torus with a plane parallel to the torus axis. The curve may be either a single closed curve or the union of two disjoint components. The curve may also have inflection points (flexes) or not, according to the distance from the plane to the torus axis.

In this paper, we study the existence of flexes for general spirics, using computations of Hessians. If the curve $\mathscr{C}$ is given by an implicit equation of the form $F(x,y) = 0$, then its Hessian curve is given by the vanishing points of the determinant $\det \frac{\partial d^2 F}{\partial x^i \, \partial y^j}$, for $i + j = 2$ (necessary condition, but not sufficient). The flexes of $\mathscr{C}$ are intersections of the curve with its Hessian curve ([4], [5]). The GeoGebra system is used for dynamical visualization, and a the Maple software is used for automated study of the curves and their intersections. In particular, it must be noted that the equations involved here are of high degree (not less than 4).

References

[1] Cieślak. W., Miernowski, A and Mozgawa, W.: *Isoptics of a closed strictly convex curve*, in Global Differential Geometry and Global Analysis, Lecture Notes in Mathematics 1481, 28-35 (1991).

[2] Dana-Picard, Th., Mann, G. and Zehavi, N.: *From conic intersections to toric intersections: the case of the isoptic curves of an ellipse*, The Montana Mathematical Enthusiast 9 (1), 59-76. Available: http://www.math.umt.edu/TMME/vol9no1and2/index.html (2011).

[3] Dana-Picard, Th., Mann, G. and Zehavi, N.: *Bisoptic curves of a hyperbola*, International Journal of Mathematical Education in Science and Technology **45 (5)**, pp. 762-781 (2014).

[4] Fulton, W.: Algebraic Curves: An Introduction to Algebraic Geometry, Amsterdam: W. A. Benjamin (1969).

[5] Kirwan, F.: Complex Algebraic Curves, London Mathematical Society Students Texts 23 (1992).

[7] Miernowski, A. and Mozgawa, W.: *On some geometric condition for convexity of isoptics*, Rend. Sem. Mat. Univ. Pol. Torino, 55, 2, 93-98 (1997).

[8] Szałkowski D.: *Isoptics of open rosettes*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A 59, 119-128 (2005)

[9] Wassenaar J.: Plane spiric curve, 2003. Available: http://www.2dcurves.com/quartic/quarticsp.html