On the closest distance between a point and a convex body

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In this lecture we fix a strictly convex body in the plane and a point in its exterior. We investigate the following problem: find the point M' on the boundary of the fixed convex body C realizing the minimal distance to the given point M.

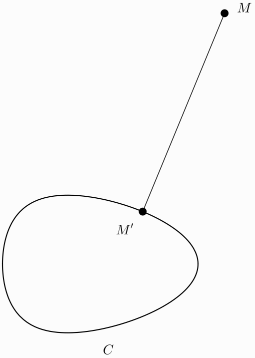


Figure: Finding the closest point M'

Introductory facts

Let C be a plane closed strictly convex curve that the origin O of the coordinate system lies in the region bounded by C. We denote by p the support function of C with respect to the origin O, where $p\colon \mathbb{R} \to \mathbb{R}$, $p(t) = \sup_{z\in C}\langle z, e^{it}\rangle$. The support function p is differentiable and the parametrization of C in terms of this function is given by

$$\begin{aligned} z(t) &= p(t)e^{it} + p'(t)ie^{it} \\ &= p(t)\cos t - p'(t)\sin t + i\left[p(t)\sin t + p'(t)\cos t\right]. \end{aligned} \tag{1}$$

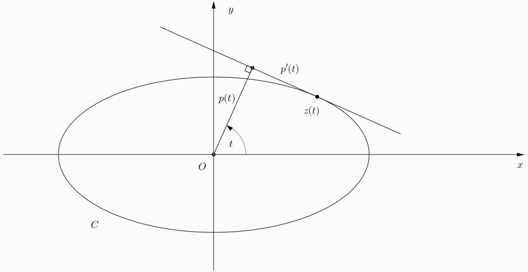


Figure: Illustration of the support function $p(t)$ of the curve C with respect to the origin O, its derivative $p'(t)$ and the point $z(t)$

Introductory facts

We assume that $z(0)$ lies in the first quadrant. We find an equation of support line to C passing through a given point $(b, 0)$, where $b > p(0)$. We introduce the notations as on the figure below.

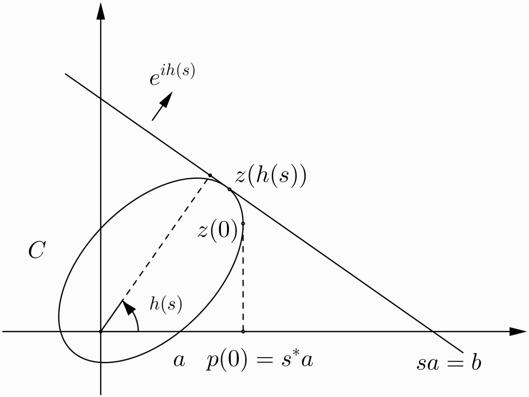


Figure: Quantities a, $h(s)$, s^*a, sa

Introductory facts

We will consider h as a function of the variable $s \in (s^*, +\infty)$ with values in the interval $\left(0, \frac{\pi}{2}\right)$. For a fixed s we have

$$z(h(s)) + rie^{ih(s)} = sa$$

for some r. Hence we get

$$p(h(s)) = as\cos h(s).$$

Let

$$f(u) = \frac{p(u)}{a\cos u} \qquad \text{for } u \in \left(0, \frac{\pi}{2}\right).$$

It is easy to see that

$$\begin{cases} f'(u) = \frac{\operatorname{Im} z(u)}{a\cos^2 u}, \\ f(0) = \frac{p(0)}{a}, \qquad f'(0) = \frac{p'(0)}{a}. \end{cases}$$

Introductory facts

Our assumptions imply that $z(u)$ for $u \in \left(0, \frac{\pi}{2}\right)$ lies in the upper half-plane. Thus, we have $\operatorname{Im} z(u) > 0$ for $u \in \left(0, \frac{\pi}{2}\right)$, and f is a strictly increasing function.

We note that the condition $p(h(s)) = as \cos h(s)$ can be rewritten in the form $f \circ h = \mathrm{id}$. The function f is invertible, so we have

$$h = f^{-1}.$$

If $b = sa$ then our support line has the following equation

$$x + y \tan f^{-1}\left(\frac{b}{a}\right) - b = 0.$$

Example

Let $r > 8$ and $p(t) = r - \cos 3t$. We have $p(t) > 0$, $p(t + 2\pi) = p(t)$ and $p(t) + p''(t) = r + 8\cos 3t > 0$. Thus p is a support function of some strictly convex curve C.

Introductory facts

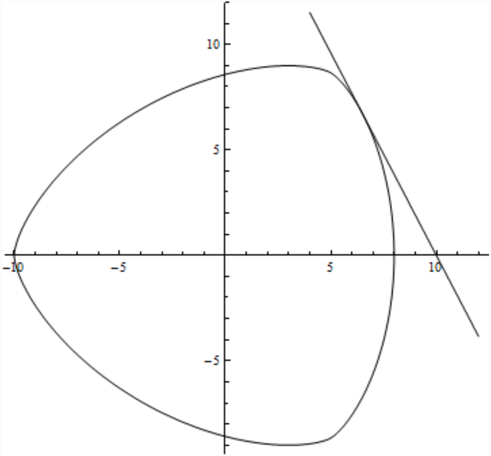


Figure: $p(t) = 9 - \cos 3t$

Introductory facts

Example continuation

We note that $z(0) = r - 1$, so $a = r - 1$ and $z(u)$ for $u \in (0, \frac{\pi}{2})$ lies in the upper half-plane. The function f in our case has the form

$$f(u) = \frac{r - \cos 3u}{(r-1)\cos u}.$$

We find the inverse function f^{-1} to f. If we set

$$s = \frac{r - \cos 3u}{(r-1)\cos u}$$

and

$$v = \cos u$$

then we have the following equation of the third degree

$$4v^3 + (-3 + (r-1)s)v - r = 0.$$

Introductory facts

Example continuation

This equation has exactly one solution $v(s)$ in the interval $(0,1)$, namely

$$v(s) = \frac{-3^{2/3}rs + 3^{2/3}(s+3) + \sqrt[3]{3}\left(9r + \sqrt{3}\sqrt{((r-1)s-3)^3 + 27r^2}\right)^{2/3}}{6\sqrt[3]{9r + \sqrt{3}\sqrt{((r-1)s-3)^3 + 27r^2}}}.$$

Thus the inverse function to f is given by

$$\tan^2 f^{-1}(s) = \frac{1}{v(s)^2} - 1$$

so our support line of C passing through a point $(b,0)$ has the following equation

$$x + y\sqrt{\frac{1}{v\left(\frac{b}{r-1}\right)^2} - 1} - b = 0.$$

Main result

We begin by an auxiliary lemma.

Lemma

Let C be a strictly convex curve given by (1) and $a = z(t^*) > 0$. If C satisfies the condition

$$\operatorname{Im} z(0) < 0, \tag{2}$$

then the function $Q\colon (0, t^*) \to \mathbb{R}$ given by the formula

$$Q(u) = -\frac{p'(u)}{a \sin u} \tag{3}$$

is positive-valued and strictly decreasing. If C satisfies the condition $\operatorname{Im} z(0) > 0$, then the function $Q^0\colon (t^*, 2\pi) \to \mathbb{R}$ given by the formula

$$Q^0(v) = -\frac{p'(v)}{a \sin v}$$

is positive-valued and strictly increasing.

Main result

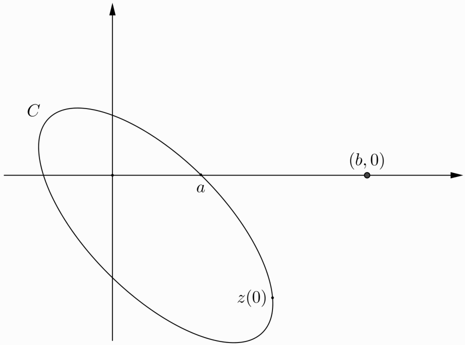


Figure: $\operatorname{Im} z(0) < 0$ and $b > a$

Proof. We may assume that $a = 1$ and we prove the first part of the lemma since the second one can be proved similarly. Note that

$$Q(u)\cos u = p(u) - \frac{\operatorname{Im} z(u)}{\sin u}.$$

If $u \in (0, t^*)$, then $\operatorname{Im} z(u) < 0$ and Q is a positive-valued function. Now, we prove that Q is strictly decreasing. Let $r, s \in (0, t^*)$ and $r < s$. Obviously, we have

$$\begin{cases} p(r)\cos r - p'(r)\sin r > p(s)\cos s - p'(s)\sin s \\ -p(r)\sin r - p'(r)\cos r > -p(s)\sin s - p'(s)\cos s. \end{cases} \tag{4}$$

It follows immediately from (4) that

$$p(r)\sin(s-r) - p'(r)\cos(s-r) > -p'(s) \tag{5}$$

for arbitrary $r, s \in (0, t^*)$ such that $r < s$.

Now, let's assume that there exist $\hat{s}, \hat{r} \in (0, t^*)$ such that $\hat{r} < \hat{s}$ and $Q(\hat{r}) \leq Q(\hat{s})$, i.e.

$$p'(\hat{r}) \sin \hat{r} \cos \hat{u} + p'(\hat{r}) \cos \hat{r} \sin \hat{u} \geq p'(\hat{s}) \sin \hat{r}, \tag{6}$$

where $\hat{u} = \hat{s} - \hat{r} > 0$. On the other hand from (5) we have

$$p(\hat{r}) \sin \hat{u} - p'(\hat{r}) \cos \hat{u} > -p'(\hat{s}).$$

Multiplying both sides of the above inequality by $\sin \hat{r}$ and then adding to (6) we get

$$[p(\hat{r}) \sin \hat{r} + p'(\hat{r}) \cos \hat{r}] \sin \hat{u} > 0,$$

what means that $\operatorname{Im} z(\hat{r}) > 0$ and we get a contradiction. □

Now, we use this lemma to show our main result.

Main result

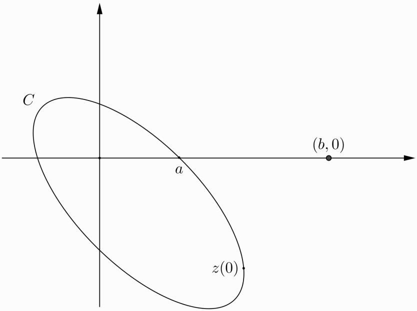


Figure: $\operatorname{Im} z(0) < 0$ and $b > a$

Theorem

Let C be a strictly convex curve given by (1) and $a = z(t^*) > 0$. If $b > a$ and $\operatorname{Im} z(0) < 0$ then the point $z\left(Q^{-1}\left(\frac{b}{a}\right)\right)$, where $Q(u) = -\frac{p'(u)}{a \sin u}$, realizes the shortest distance between $(b, 0)$ and C.

Proof.
Let C satisfy the condition $\operatorname{Im} z(0) < 0$ and $b > a$. We find a point on C realizing the shortest distance between the point $(b, 0)$ and C. We use notations as on the figure below.

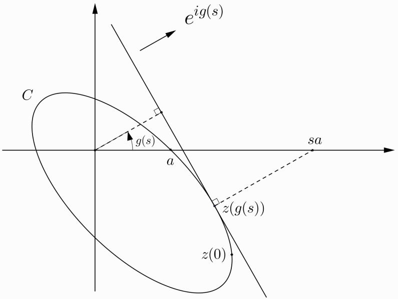


Figure: $b = sa$ and the definition of the function $g(s)$

g is a function of the variable $s \in (1, +\infty)$ with values in $(0, t^*)$. It is clear that the vectors $e^{ig(s)}$ and $z(g(s)) - sa$ have to be collinear, i.e.

$$z(g(s)) - sa = re^{ig(s)}$$

for some $r \in \mathbb{R}$. Hence we get

$$-p'(g(s)) = sa \sin g(s). \tag{7}$$

This formula can be rewritten in the following form

$$Q \circ g = \mathrm{id}\,.$$

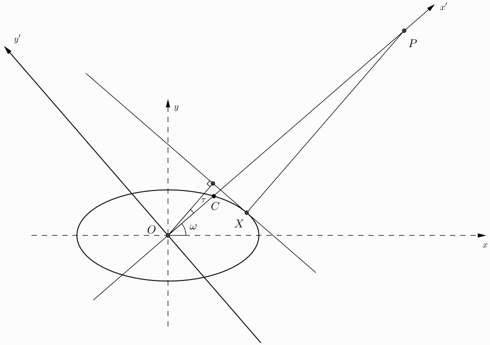
In view of the Lemma above we have $g = Q^{-1}$. Thus we proved our main theorem. $\square$

An example

In this example we illustrate our above method and show its simplicity. We will determine the distance from the point $P\left(3\sqrt{3}, \frac{9}{2}\right)$ to the ellipse $\frac{x^2}{4} + y^2 = 1$. Such ellipse has the support function in Oxy coordinates given by the formula

$$p(t) = \sqrt{4\cos^2 t + \sin^2 t}$$

and the situation is illustrated on the figure below.



An example

Then the formula (1) becomes

$$z(t) = \frac{12 + 4\cos 2t + \sin 2t}{4\sqrt{4\cos t + \sin t}} + \frac{3 - \cos 2t + 4\sin 2t}{4\sqrt{4\cos t + \sin t}} i.$$

Note that in the notations of the above figure we have that the point C has in the system Oxy the coordinates $\left(1, \frac{\sqrt{3}}{2}\right)$. We rotate the coordinate system counter clockwise about the angle ω between the radius vector of the point P and the axis Ox. Using the coordinates of P we get immediately that

$$\tan\omega = \frac{\sqrt{3}}{2}$$

thus the equation of our ellipse in new coordinates $Ox'y'$ becomes

$$16\left(x'\right)^2 + 12\sqrt{3}x'y' + 19\left(y'\right)^2 = 28.$$

Evidently, the points C and P lie now on the Ox' axis and have the new coordinates $\left(\frac{\sqrt{7}}{2}, 0\right)$ and $\left(\frac{3\sqrt{21}}{2}, 0\right)$, respectively. Hence $a = |OC| = \frac{\sqrt{7}}{2}$ and $b = |OP| = \frac{3\sqrt{21}}{2}$.

An example

The support function $q(t)$ of our ellipse in new coordinate system has the form

$$q(t) = p(t+\omega) = \sqrt{4\cos^2(t+\omega) + \sin^2(t+\omega)}.$$

Next from the figure above we read that

$$\tan\left(t^* + \omega\right) = 2\sqrt{3}$$

hence

$$\tan t^* = \frac{8\sqrt{3}}{3} \approx 4.6188\ldots$$

and $t^* \approx 1.35758$. Thus from our Lemma we get that the function $Q(u) = -\frac{q'(u)}{a\sin u}$ is positive-valued and strictly decreasing on interval $(0, t^*)$. So we need to determine $u = Q^{-1}\left(\frac{b}{a}\right)$. We solve the equation $Q(u) = \frac{b}{a}$ i.e.

$$-\frac{q'(u)}{a\sin u} = \frac{b}{a}$$

or

$$p'(u+\omega) = -b\sin u.$$

An example

Inserting all necessary data, $\cos\omega = \frac{2}{\sqrt{7}}$ and $\sin\omega = \frac{\sqrt{3}}{\sqrt{7}}$ and substituting $x = \tan u$ we arrive to the quartic equation

$$2304x^4 - 1748\sqrt{3}x^3 + 2885x^2 - 16\sqrt{3}x - 48 = 0.$$

Letting $\Omega(x) = 2304x^4 - 1748\sqrt{3}x^3 + 2885x^2 - 16\sqrt{3}x - 48$ it is easy to see that $\Omega(0) < 0$ and $\Omega(1) > 0$, moreover $1 = \tan\frac{\pi}{4} < \tan t^*$ and Q is strictly decreasing thus there exists exactly one solution $x_0 \in (0, t^*)$ which gives the closest point on the ellipse to P. One can check directly that $x_0 = \frac{1}{4\sqrt{3}}$. Thus in Oxy coordinates we have that $X = z(\omega + \arctan x_0) = \sqrt{3} + \frac{1}{2}i$. The point $\left(\sqrt{3}, \frac{1}{2}\right)$ is the searched point realizing the claimed minimum, as we see on the figure above.

Computer approximation

In general it is not easy to obtain the inverse of the function Q in our Theorem but we can approximate the inverse, what gives us the possibility of finding approximation that realizes the shortest distance between a given point and a strictly convex curve.

An algorithm deploying ideas previously introduced in this talk could be divided into two parts, the first one (part A) is to be done once for a given convex set, the second (part B) – once for a given point.

Computer approximation

A-1. Let $p(t)$ be the support function of our convex set bounded by C. If $p(t)$ cannot be obtained as an analytical expression for some reasons it can be approximated in various ways. For example one can set large integer M, set $h = 2\pi/M$ and for all $t \in \{0, h, 2h, \ldots, (M-1)h\}$ calculate

$$p(t) = \max_{(x,y)\in C}(x\cos t + y\sin t) \tag{8}$$

How easily the value in (8) can be obtained depends on the nature of C, in most practical cases, like C being a set of Bézier curves, it is straightforward. For example if a Bézier curve is cubic, then the first derivative (with respect to s) of

$$x(s)\cos t + y(s)\sin t, \quad s \in (0,1)$$

is a polynomial in s of the second order.
But in general finding the support function (or its approximation) is beyond the scope of this talk, and we just assume it is calculated. Once for a given C.

Computer approximation

A-2. Let $p'(t)$ be a derivative of $p(t)$. If we do not have an analytical expression for $p(t)$, we can, for $t \in \{0, h, 2h, \dots, (M-1)h\}$ approximate $p'(t)$ as

$$\frac{p(t+h) - p(t-h)}{2h}.$$

Again, obtaining $p'(t)$ is done once for given C.

A-3. Having p and p' we may find all t^* for all $t \in \{0, h, 2h, \dots, (M-1)h\}$, such that

$$\arg(z(t^*)) \approx t, \qquad t, t^* \in \{0, h, 2h, \dots, (M-1)h\}.$$

Having these function (or their approximations) prepared, we may proceed with our main algorithm.

Computer approximation

Suppose we have a point $be^{i\alpha}$, outside of our C, we want to find the closest point on C.

B-1. Obtain t^* for $t \approx \alpha$ (prepared in *A-3*)

B-2. If $p'(\alpha) = 0$ (or, in other words, $t^* = \alpha$) then our optimal point is $p(\alpha)e^{i\alpha}$, the algorithm stops,

B-3. If $p'(\alpha) < 0$ then let $L = \alpha$, let $R = t^*$ (and if R happened to be smaller than L then advance R by 2π

B-4. If $p'(\alpha) > 0$ then let $L = t^*$, let $R = \alpha$ (and if L happened to be greater than R then reduce L by 2π

B-5. Now solve equation

$$\frac{-p'(u)}{\sin(u - \alpha)} = b, \qquad u \in (L, R). \tag{9}$$

Our Lemma guarantees uniqueness of the solution, and since the left side of (9) is strictly monotonic, so it can be solved by simple numerical methods, for example by bisection.

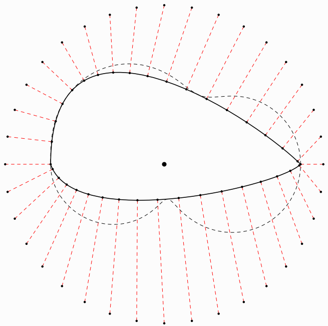
B-6. Due to our Theorem the optimal point is

$$z(u) = p(u)e^{iu} + p'(u)ie^{it}.$$

Computer approximation

Example

Let us consider a curve described by two cubic Bézier curves illustrated on the figure below. The first curve is defined by points $(-5, 0)$, $(-5, 8)$, $(4, 2)$, $(6, 0)$. The second one is defined by $(-5, 0)$, $(-4, -3)$, $(5 - 1)$, $(6, 0)$. Curves are illustrated with solid lines. A dashed line near the curves is a pedal line $p(t)e^{it}$. Then we have 36 points of the form $7 \cdot e^{i2\pi/36}$ and for each we draw its closest point on C, connected with dashed segment.



Computer approximation

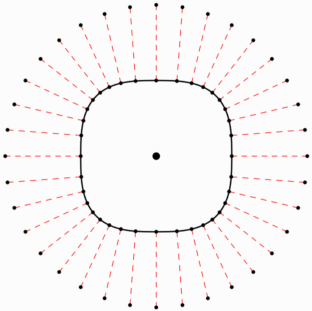
All calculations for drawing the picture were done exactly with the described algorithm.

Example

Another example is a convex set bounded by a curve

$$|x|^3 + |y|^3 = 1.$$

Again, applying the above algorithms gives us the closest points to particular 36 points of the form $2 \cdot e^{i2\pi/36}$, illustrated on the figure below.



Thank you for your attention!