Constructing Linkages for Drawing Plane Curves

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We describe an application of computer algebra to the construction of mechanisms with certain prescribed properties. For this purpose, we have developed the package **PlanarLinkages** in Mathematica; it provides commands for constructing and visualizing planar linkages that draw a prescribed algebraic curve. The construction procedure is based on so-called motion polynomials; their basic arithmetic and a factorization algorithm is also provided by the package.

A *linkage* is a mechanical device consisting of rigid bodies (called *links*) that are connected by *joints*. We restrict our attention to *planar linkages*, i.e., to linkages all of whose links move in parallel planes. Moreover, we consider only *rotational joints*, i.e., we don't allow *prismatic joints*. We say that a linkage has *mobility one*, if it has only one degree of freedom; if we move a linkage of mobility one, the trace of any point located on one of the links yields a bounded curve in the plane.

The problem of constructing a planar linkage that draws a finite segment of a given algebraic curve was first addressed and solved in full generality by Kempe [2]. While his construction is very elegant in theory, it yields quite complicated linkages in practice. In a recent article [1], the symbolic computation group in Linz designed a novel algorithm for basically the same problem. The advantage of the new algorithm is that it yields much simpler linkages: the number of links and joints is only linear in the degree of the curve. Moreover, it allows for a simple collision detection, which for general linkages is a very hard problem. The drawback of our method is that it is only applicable to bounded rational curves, i.e., to curves that are parametrizable by rational functions and that are contained in some disk of finite radius.

A *motion* is a one-dimensional family of direct isometries (i.e., translations and rotations). We denote by SE_2 the special Euclidean group, which is the set of direct isometries in the plane with composition as the group operation. For a convenient treatment in a computer algebra system, we encode direct isometries as elements of the noncommutative $\mathbb{R}$-algebra $\mathbb{K}$ of *dual complex numbers*:

$$\mathbb{K} = \mathbb{C}[\eta]\,/\,(\eta^2, i\eta + \eta i).$$

Its elements are of the form $z + \eta w$ with complex numbers $z, w \in \mathbb{C}$, and according to the defining relations, which can be seen as rewriting rules, they are multiplied as follows:

$$(z_1 + \eta w_1) \cdot (z_2 + \eta w_2) = z_1 z_2 + \eta\,(\overline{z_1} w_2 + z_2 w_1). \tag{1}$$

By defining on $\mathbb{K}$ the equivalence relation

$$k_1 \sim k_2 \;:\Longleftrightarrow\; k_1 = \alpha k_2 \text{ for some } \alpha \in \mathbb{R} \setminus \{0\}, \tag{2}$$

we can show that the multiplicative group

$$\{z + \eta w \in \mathbb{K} \mid z \neq 0\} / \sim$$

is isomorphic to SE_2. A univariate polynomial in $\mathbb{K}[t]$ then gives rise to a one-dimensional family of direct isometries and is therefore called a *motion polynomial*. Motions that can be represented in this way are called *rational motions*. Our algorithm takes as input a motion polynomial and outputs a planar linkage of mobility one realizing the corresponding rational motion. This task is slightly more general than drawing a rational curve, since also the orientation of the end effector can be taken into account.

A motion polyomial $P = Z + \eta W \in \mathbb{K}[t]$ is called *bounded* if the complex polynomial $Z \in \mathbb{C}[t]$ does not have any real roots; the connection to the boundedness of the corresponding curve (the orbit of the origin) is established by the fact that Z appears as the denominator of its parametrization.

In order to construct a linkage that realizes the motion described by $P(t)$, we want to decompose it into simpler motions, namely into revolutions; these correspond exactly to motions that can be realized by a single (rotational) joint. We find [1, Lemma 4.3] that each linear motion polynomial, whose orbits are bounded, represents a revolute motion. Therefore, the desired decomposition is obtained by a factorization of P into linear polynomials; we present an algorithm for this task.

The factorization allows us to construct a linkage, in the form of an open chain, whose links can move according to the revolutions represented by the linear factors. Since such a linkage has many degrees of freedom, we need to constrain its mobility. This is done by adding more links and joints, which is achieved by an iteration of the so-called flip procedure [1, Sections 6–7].

References

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